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### ► To cite this version:

Lubomira Balkova, Edita Pelantová, Wolfgang Steiner. Sequences with constant number of return words. Monatshefte für Mathematik, 2008, 155 (3-4), pp.251-263. 10.1007/s00605-008-0001-2 . hal-00089863v4

**HAL Id: hal-00089863**

**<https://hal.science/hal-00089863v4>**

Submitted on 27 Sep 2007

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# SEQUENCES WITH CONSTANT NUMBER OF RETURN WORDS

LUBOMÍRA BALKOVÁ, EDITA PELANTOVÁ, AND WOLFGANG STEINER

**ABSTRACT.** An infinite word has the property  $R_m$  if every factor has exactly  $m$  return words. Vuillon showed that  $R_2$  characterizes Sturmian words. We prove that a word satisfies  $R_m$  if its complexity function is  $(m-1)n+1$  and if it contains no weak bispecial factor. These conditions are necessary for  $m=3$ , whereas for  $m=4$  the complexity function need not be  $3n+1$ . New examples of words satisfying  $R_m$  are given by words related to digital expansions in real bases.

## 1. INTRODUCTION

Recently, return words have been intensively studied in (symbolic) dynamical systems, combinatorics on words and number theory. Roughly speaking, for a given factor  $w$  of an infinite word  $u$ , a return word of  $w$  is a word between two successive occurrences of the factor  $w$ . This can be seen as a symbolic version of the first return map in a dynamical system. This notion was introduced by Durand [5] to give a nice characterization of primitive substitutive sequences. A slightly different notion of return words was used by Ferenczi, Mauduit and Nogueira [8].

Sturmian words are aperiodic words over a bilateral alphabet with the lowest possible factor complexity; they were defined by Morse and Hedlund [11]. Using return words, Vuillon [13] found a new equivalent definition of Sturmian words. He showed that an infinite word  $u$  over a bilateral alphabet is Sturmian if and only if any factor of  $u$  has exactly two return words. A short proof of this fact is given in Section 5.

A natural generalization of Sturmian words to  $m$ -letter alphabets is constituted by infinite words with every factor having exactly  $m$  return words. This property is called  $R_m$ . It covers other generalizations of Sturmian words: Justin and Vuillon [10] proved that Arnoux-Rauzy words of order  $m$  satisfy  $R_m$ , Vuillon [14] proved this property for words coding regular  $m$ -interval exchange transformations.

The factor complexity, i.e., the number of different factors of length  $n$ , of the two classes of words with property  $R_m$  in the preceding paragraph is  $(m-1)n+1$  for all  $n \geq 0$ . Vuillon [14] observed that this condition is not sufficient to describe words satisfying  $R_m$ ,  $m \geq 3$ : the fixed point of a certain recoding of the Chacon substitution, which has complexity  $2n+1$  by Ferenczi [6], has factors with more than 3 return words.

A deeper inspection of the two classes of words with property  $R_m$  shows that not only the first difference of complexity is constant, but also that the bilateral order of every factor (see Cassaigne [4] and Section 4) is zero. We show that this condition is indeed sufficient to have the property  $R_m$ , and provide a less known class of words satisfying this condition. If a word satisfies  $R_3$ , then we can show that no factor is weak bispecial, i.e., no factor has negative bilateral order. Therefore the words with  $R_3$  are characterized by complexity  $2n+1$  and the absence of weak bispecial factors.

In Section 6.1, we provide a word satisfying  $R_4$  with an even number of factors of every positive length (containing infinitely many weak bispecial factors). Therefore words satisfying  $R_m$  do not necessarily have complexity  $(m-1)n+1$ , and it is an open question whether there exists a nice characterization of words satisfying  $R_m$  for  $m \geq 4$ .

We conclude the article by exhibiting a large class of purely substitutive words satisfying  $R_m$ . Every word in this class codes the sequence of distances between consecutive  $\beta$ -integers for some real number  $\beta > 1$ .

In this article we focus only on the number of return words corresponding to a given factor of an infinite word. We do not study the ordering of return words in the infinite word, i.e., we do not study derivated sequences (see [5] for the precise definition). Let us just mention here that a derivated sequence of a word with property  $R_m$  is again a word satisfying  $R_m$ . A description of derivated sequences of Sturmian words can be found in [1].

## 2. BASIC DEFINITIONS

An *alphabet*  $\mathcal{A}$  is a finite set of symbols called *letters*. A (possibly empty) concatenation of letters is a *word*. The set  $\mathcal{A}^*$  of all finite words provided with the operation of concatenation is a free monoid. The *length* of a word  $w$  is denoted by  $|w|$ . A finite word  $w$  is called a *factor* (or *subword*) of the (finite or right infinite) word  $u$  if there exist a finite word  $v$  and a word  $v'$  such that  $u = v w v'$ . The word  $w$  is a *prefix* of  $u$  if  $v$  is the empty word. Analogously,  $w$  is a *suffix* of  $u$  if  $v'$  is the empty word. A concatenation of  $k$  words  $w$  will be denoted by  $w^k$ .

The *language*  $\mathcal{L}(u)$  is the set of all factors of the word  $u$ , and  $\mathcal{L}_n(u)$  is the set of all factors of  $u$  of length  $n$ . Let  $w$  be a factor of an infinite word  $u$  and let  $a, b \in \mathcal{A}$ . If  $aw$  is a factor of  $u$ , then we call  $a$  a *left extension* of  $w$ . Analogously, we call  $b$  a *right extension* of  $w$  if  $wb \in \mathcal{L}(u)$ . We will denote by  $\mathcal{E}_\ell(w)$  the set of all left extensions of  $w$ , and by  $\mathcal{E}_r(w)$  the set of right extensions. A factor  $w$  is *left special* if  $\#\mathcal{E}_\ell(w) \geq 2$ , *right special* if  $\#\mathcal{E}_r(w) \geq 2$  and *bispecial* if  $w$  is both left special and right special.

Let  $w$  be a factor of an infinite word  $u = u_0 u_1 \dots$  (with  $u_j \in \mathcal{A}$ ),  $|w| = \ell$ . An integer  $j$  is called an *occurrence* of  $w$  in  $u$  if  $u_j u_{j+1} \dots u_{j+\ell-1} = w$ . Let  $j, k$ ,  $j < k$ , be successive occurrences of  $w$ . Then  $u_j u_{j+1} \dots u_{k-1}$  is a *return word* of  $w$ . The set of all return words of  $w$  is denoted by  $\mathcal{R}(w)$ ,

$$\mathcal{R}(w) = \{u_j u_{j+1} \dots u_{k-1} \mid j, k \text{ being successive occurrences of } w \text{ in } u\}.$$

If  $v$  is a return word of  $w$ , then the word  $vw$  is called *complete return word*.

An infinite word is *recurrent* if any of its factors occurs infinitely often or, equivalently, if any of its factors occurs at least twice. It is *uniformly recurrent* if, for any  $n \in \mathbb{N}$ , every sufficiently long factor contains all factors of length  $n$ . It is not difficult to see that a recurrent word on a finite alphabet is uniformly recurrent if and only if the set of return words of any factor is finite.

The variability of local configurations in  $u$  is expressed by the *factor complexity function* (or simply *complexity*)  $C(n) = \#\mathcal{L}_n(u)$ . It is well known that a word  $u$  is aperiodic if and only if  $C(n) \geq n + 1$  for all  $n \in \mathbb{N}$ . Infinite aperiodic words with the minimal complexity  $C(n) = n + 1$  for all  $n \in \mathbb{N}$  are called *Sturmian words*. These words have been studied extensively, and several equivalent definitions of Sturmian words can be found in Berstel [3].

## 3. SIMPLE FACTS FOR RETURN WORDS

**3.1. Restriction to bispecial factors.** If a factor  $w$  is not right special, i.e., if it has a unique right extension  $b \in \mathcal{A}$ , then the sets of occurrences of  $w$  and  $wb$  coincide, and

$$\mathcal{R}(w) = \mathcal{R}(wb).$$

If a factor  $w$  has a unique left extension  $a \in \mathcal{A}$ , then  $j \geq 1$  is an occurrence of  $w$  in the infinite word  $u$  if and only if  $j - 1$  is an occurrence of  $bw$ . This statement does not hold for  $j = 0$ . Nevertheless, if  $u$  is a recurrent infinite word, then the set of return words of  $w$  stays the same no matter whether we include the return word corresponding to the prefix  $w$  of  $u$  or not. Consequently, we have

$$\mathcal{R}(aw) = a\mathcal{R}(w)a^{-1} = \{ava^{-1} \mid v \in \mathcal{R}(w)\},$$

where  $ava^{-1}$  means that the word  $v$  is prolonged to the left by the letter  $a$  and it is shortened from the right by erasing the letter  $a$  (which is always the suffix of  $v$  for  $v \in \mathcal{R}(w)$ ).

For an aperiodic uniformly recurrent infinite word  $u$ , each factor  $w$  can be extended to the left and to the right to a bispecial factor. To describe the cardinality and the structure of  $\mathcal{R}(w)$  for arbitrary  $w$ , it suffices therefore to consider bispecial factors  $w$ .

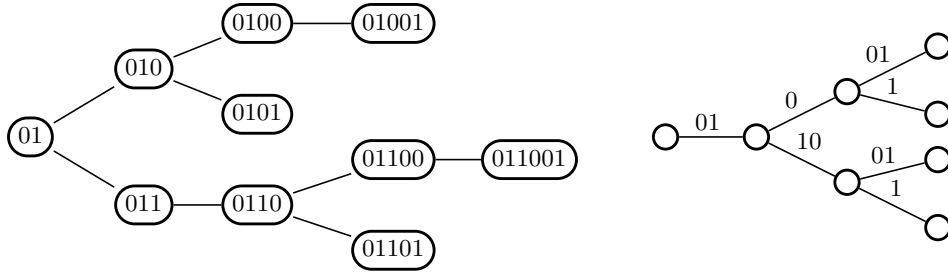


FIGURE 1. The tree of return words of 01 in the Thue-Morse sequence and its trie representation.

**3.2. Tree of return words.** It is convenient to consider a tree (or trie) constructed in the following way: Label the root with a factor  $w$ , and attach  $\#\mathcal{E}_r(w)$  children, with labels  $wb$ ,  $b \in \mathcal{E}_r(w)$ . Repeat this recursively with every node labeled by  $v$ , except if  $w$  is a suffix of  $v$ . If  $u$  is uniformly recurrent, then this algorithm stops, and it is easy to see that the labels of the leaves of this tree are exactly the complete return words of  $w$ . Therefore we have

$$(1) \quad \#\mathcal{R}(w) = \#\{\text{leaves}\} = 1 + \sum_{\text{non-leaves } v} (\#\mathcal{E}_r(v) - 1).$$

In particular, if  $w$  is the unique right special factor of its length, then  $\#\mathcal{R}(w) = \#\mathcal{E}_r(w)$ .

A similar construction can be done with left extensions, yielding similar formulae. Since we can restrict our attention to bispecial factors  $w$  by Section 3.1, we obtain the following proposition.

**Proposition 3.1.** *Let  $u$  be a recurrent word and  $m \in \mathbb{N}$ . Suppose that for every  $n \in \mathbb{N}$  at least one of the following conditions is satisfied:*

- *There is a unique left special factor  $w \in \mathcal{L}_n(u)$ , and  $\#\mathcal{E}_\ell(w) = m$ .*
- *There is a unique right special factor  $w \in \mathcal{L}_n(u)$ , and  $\#\mathcal{E}_r(w) = m$ .*

*Then  $u$  satisfies property  $R_m$ , i.e., every factor has exactly  $m$  return words.*

Recall that Arnoux-Rauzy words of order  $m$  are defined as uniformly recurrent infinite words which have for every  $n \in \mathbb{N}$  exactly one right special factor  $w$  of length  $n$  with  $\#\mathcal{E}_r(w) = m$  and exactly one left special factor  $w$  of length  $n$  with  $\#\mathcal{E}_\ell(w) = m$ . They are also called strict episturmian words. It is easy to see that Sturmian words are recurrent, and we obtain the following corollary to Proposition 3.1.

**Corollary 3.2.** *Arnoux-Rauzy words of order  $m$  satisfy  $R_m$ , in particular Sturmian words satisfy  $R_2$ .*

#### 4. SUFFICIENT CONDITIONS FOR PROPERTY $R_m$

This section is devoted to sufficient conditions for a word  $u$  having the property  $R_m$ , but we mention first two evident necessary conditions.

The alphabet  $\mathcal{A}$  of  $u$  must have  $m$  letters since the occurrences of the empty word are all integers  $n \geq 0$ , and its return words are therefore all letters  $u_n$ . Furthermore,  $u$  must be uniformly recurrent since every factor has a return word and only finitely many of them.

An important role in our further considerations is played by weak bispecial factors.

**Definition 4.1.** *A factor  $w$  of a recurrent word is weak bispecial if  $B(w) < 0$ , where*

$$B(w) = \#\{awb \in \mathcal{L}(u) \mid a, b \in \mathcal{A}\} - \#\mathcal{E}_\ell(w) - \#\mathcal{E}_r(w) + 1$$

*is the bilateral order of  $w$ .*

Since  $\#\{awb \in \mathcal{L}(u) \mid a, b \in \mathcal{A}\} = \sum_{a \in \mathcal{E}_\ell(w)} \#\mathcal{E}_r(aw) = \sum_{b \in \mathcal{E}_r(w)} \#\mathcal{E}_\ell(wb)$ , the inequality  $B(w) < 0$  is equivalent to

$$\sum_{a \in \mathcal{E}_\ell(w)} (\#\mathcal{E}_r(aw) - 1) < \#\mathcal{E}_r(w) - 1$$

and to

$$\sum_{b \in \mathcal{E}_r(w)} (\#\mathcal{E}_\ell(wb) - 1) < \#\mathcal{E}_\ell(w) - 1.$$

The bilateral order was defined by Cassaigne [4] in order to calculate the second complexity difference. If we set  $\Delta C(n) = C(n+1) - C(n)$ , then we have

$$\Delta C(n) = \sum_{w \in \mathcal{L}_n(u)} (\#\mathcal{E}_\ell(w) - 1) = \sum_{w \in \mathcal{L}_n(u)} (\#\mathcal{E}_r(w) - 1)$$

and therefore

$$\begin{aligned} \Delta C(n+1) - \Delta C(n) &= \sum_{w \in \mathcal{L}_n(u)} \sum_{a \in \mathcal{E}_\ell(w)} (\#\mathcal{E}_r(aw) - 1) - \sum_{w \in \mathcal{L}_n(u)} (\#\mathcal{E}_r(w) - 1) \\ &= \sum_{w \in \mathcal{L}_n(u)} (\#\{awb \in \mathcal{L}(u) \mid a, b \in \mathcal{A}\} - \#\mathcal{E}_\ell(w) - \#\mathcal{E}_r(w) + 1) = \sum_{w \in \mathcal{L}_n(u)} B(w). \end{aligned}$$

If  $B(w) = 0$  for all factors  $w$ , then the first complexity difference is constant. If no factor is weak bispecial, then  $\Delta C(n)$  is non-decreasing. Since  $\Delta C(0) = \#\mathcal{A} - 1$  and  $\#A = m$ , we obtain the following lemma.

**Lemma 4.2.** *If  $u$  satisfies  $R_m$  and no factor is weak bispecial, then  $\Delta C(n) \geq m - 1$  for all  $n \geq 0$ .*

The number of return words can be bounded by the following lemmas.

**Lemma 4.3.** *If  $u$  is a uniformly recurrent word with no weak bispecial factor, then*

$$\#\mathcal{R}(w) \geq 1 + \Delta C(|w|)$$

for every factor  $w \in \mathcal{L}(u)$ .

*Proof.* Let  $w \in \mathcal{L}(u)$  and denote by  $v_1, v_2, \dots, v_r$  the right special factors of length  $|w|$ . Since no factor is weak bispecial and  $u$  is uniformly recurrent, every  $v_j$  can be extended to the left without decreasing the total amount of “right branching” until  $w$  is reached. More precisely, we have (mutually different) right special factors  $v_j^{(1)}, v_j^{(2)}, \dots, v_j^{(s_j)}$  with suffix  $v_j$ , prefix  $w$  and no other occurrence of  $w$  such that  $\#\mathcal{E}_r(v_j) - 1 \leq \sum_{i=1}^{s_j} (\#\mathcal{E}_r(v_j^{(i)}) - 1)$ . Since all  $v_j^{(i)}$  are nodes in the tree of return words and  $v_j^{(i)} \neq v_{j'}^{(i')}$  if  $(j, i) \neq (j', i')$ , we can use (1) and obtain

$$\#\mathcal{R}(w) \geq 1 + \sum_{j=1}^r \sum_{i=1}^{s_j} (\#\mathcal{E}_r(v_j^{(i)}) - 1) \geq 1 + \sum_{j=1}^r (\#\mathcal{E}_r(v_j) - 1) = 1 + \Delta C(|w|). \quad \square$$

**Lemma 4.4.** *If  $u$  has no weak bispecial factor and  $\Delta C(n) < m$  for all  $n \geq 0$ , then*

$$\#\mathcal{R}(w) \leq m$$

for every factor  $w \in \mathcal{L}(u)$ .

*Proof.* Let  $v_1, v_2, \dots, v_r$  denote the right special factors which are labels of non-leave nodes in the tree of return words of  $w$ , and  $n = \max_{1 \leq j \leq r} |v_j|$ . Since no bispecial factor is weak, every  $v_j$  can be extended to the left to factors of length  $n$  without decreasing the total amount of “right branching”. More precisely, we have (mutually different) right special factors  $v_j^{(1)}, v_j^{(2)}, \dots, v_j^{(s_j)}$  of length  $n$  with suffix  $v_j$  such that  $\#\mathcal{E}_r(v_j) - 1 \leq \sum_{i=1}^{s_j} (\#\mathcal{E}_r(v_j^{(i)}) - 1)$ . Since  $w$  occurs in  $v_j$  only as prefix, no  $v_j$  can be a proper suffix of  $v_{j'}$ . Hence we have  $v_j^{(i)} \neq v_{j'}^{(i')}$  if  $(j, i) \neq (j', i')$  and

$$\#\mathcal{R}(w) = 1 + \sum_{j=1}^r (\#\mathcal{E}_r(v_j) - 1) \leq 1 + \sum_{j=1}^r \sum_{i=1}^{s_j} (\#\mathcal{E}_r(v_j^{(i)}) - 1) \leq 1 + \Delta C(n) \leq m. \quad \square$$

For words with no weak bispecial factors, these three lemmas give a very simple characterization of the property  $R_m$ .

**Theorem 4.5.** *If  $u$  is a uniformly recurrent word with no weak bispecial factor, then it satisfies  $R_m$  if and only if  $C(n) = (m - 1)n + 1$  for all  $n \geq 0$ .*

5. PROPERTIES  $R_2$  AND  $R_3$ 

For  $m = 2$  and  $m = 3$ , we can completely characterize the words with property  $R_m$ .

**Definition 5.1.** Let  $v$  be a return word of  $w \in \mathcal{L}(u)$ . We say that return word  $v$  starts with  $b$  if  $wb$  is a prefix of the complete return word  $vw$  and that it ends with  $a$  if  $aw$  is a suffix of  $vw$ .

A right special factor  $w$  is called *maximal right special* if  $w$  is not a proper suffix of any right special factor, i.e.,  $\sum_{a \in \mathcal{E}_\ell(w)} (\#\mathcal{E}_r(aw) - 1) = 0$ . Any maximal right special factor is therefore weak bispecial.

**Lemma 5.2.** If  $w \in \mathcal{L}(u)$  is a maximal right special factor such that for any  $b \in \mathcal{E}_r(w)$  there exists a unique  $v \in \mathcal{R}(w)$  starting with  $b$ , then  $u$  is eventually periodic.

*Proof.* Denote the return words of  $w$  by  $v_1, v_2, \dots, v_r$ , where, w.l.o.g.,  $v_j$  starts with  $b_j$ , ends with  $a_j$  and  $b_{j+1}$  is the only letter in  $\mathcal{E}_r(a_j w)$  for  $1 \leq j < r$ . Then  $b_1$  is the only letter in  $\mathcal{E}_r(a_r w)$  and  $u = p(v_1 v_2 \dots v_r)^\infty$  for some prefix  $p$ .  $\square$

**Corollary 5.3.** If  $u$  satisfies  $R_2$ , then it has no maximal right special factor.

*Proof.* Assume that  $w$  is a maximal right special factor. Then the two return words of  $w$  have different starting letters, hence  $u$  is eventually periodic by Lemma 5.2 and  $\#\mathcal{R}(wa) = 1$ .  $\square$

On a binary alphabet, the notions “weak bispecial” and “maximal right special” coincide. Therefore Corollaries 3.2, 5.3 and Lemma 4.3 provide a short proof of the following theorem.

**Theorem 5.4** (Vuillon [13]). An infinite word  $u$  satisfies  $R_2$  if and only if it is Sturmian.

For words with property  $R_3$ , we need the following lemma.

**Lemma 5.5.** Let  $w$  be a weak bispecial factor with a unique  $a \in \mathcal{E}_\ell(w)$  such that more than one return word of  $w$  starts with a letter in  $\mathcal{E}_r(aw)$ , then  $\#\mathcal{R}(aw) < \#\mathcal{R}(w)$ .

*Proof.* Any return word of  $aw$  has the form  $av_1 v_2 \dots v_r a^{-1}$  for some  $r \geq 1$  and  $v_j \in \mathcal{R}(w)$ ,  $1 \leq j \leq r$ . If  $v_1$  ends with  $a$ , then  $r = 1$ . If  $v_1$  ends with  $a' \neq a$ , then the assumption of the lemma implies that there is a unique return word of  $w$  starting with a letter in  $\mathcal{E}_r(a'w)$  (and  $\#\mathcal{E}_r(a'w) = 1$ ). Therefore  $v_2$  and inductively the sequence of words  $v_2, \dots, v_r$  are completely determined by the choice of  $v_1$ . This implies that  $\#\mathcal{R}(aw)$  equals the number of return words of  $w$  starting with a letter in  $\#\mathcal{E}_r(aw)$ . Since  $w$  is weak bispecial, we have  $\#\mathcal{E}_r(aw) < \#\mathcal{E}_r(w)$  and thus  $\#\mathcal{R}(aw) < \#\mathcal{R}(w)$ .  $\square$

*Remark.* There are two cases for Lemma 5.5: Either  $aw$  is right special or there is more than one return word of  $w$  starting with the unique right extension of  $aw$ .

**Corollary 5.6.** If  $u$  satisfies  $R_3$ , then it has no weak bispecial factor.

*Proof.* Assume that  $w$  is a weak bispecial factor.

If  $\#\mathcal{E}_r(w) = 3$ , then every return word of  $w$  starts with a different letter in  $\mathcal{E}_r(w)$ . Since at most for one  $a \in \mathcal{E}_\ell(w)$ , the factor  $aw$  is right special, we obtain a contradiction to  $R_3$  by Lemma 5.2 or 5.5.

If  $\#\mathcal{E}_r(w) = 2$ , then  $\mathcal{E}_r(aw) = \{b\}$  and  $\mathcal{E}_r(a'w) = \{b'\}$ . Since, w.l.o.g., two return words of  $w$  start with  $b$  and one starts with  $b'$ , we obtain a contradiction to  $R_3$  by Lemma 5.5.  $\square$

By combining Corollary 5.6 and Theorem 4.5, we obtain the following theorem.

**Theorem 5.7.** A uniformly recurrent word  $u$  satisfies  $R_3$  if and only if  $C(n) = 2n + 1$  for all  $n \geq 0$  and  $u$  has no weak bispecial factor.

*Remarks.*

- The theorem remains true if “weak bispecial” is replaced by “maximal right special”: If  $\Delta C(n) = 2$  for all  $n \geq 0$ , then every factor  $w$  with  $\#\mathcal{E}_r(w) = 3$  is the unique right special factor of its length, and it cannot be weak bispecial. If  $\#\mathcal{E}_r(w) = 2$ , then the two notions coincide.

- By symmetry, “weak bispecial” can be replaced by “maximal left special”.
- The condition on weak bispecial factors cannot be omitted. Ferenczi [6] showed that the fixed point  $\sigma^\infty(1)$  of the substitution given by  $\sigma : 1 \mapsto 12, 2 \mapsto 312, 3 \mapsto 3312$ , a recoding of the Chacon substitution, has complexity  $2n + 1$  and it contains weak bispecial factors.

## 6. PROPERTY $R_4$

**6.1. A word with complexity  $\neq 3n + 1$ .** The following proposition shows that  $C(n)$  need not be  $(m - 1)n + 1$  for all  $n \geq 0$  if  $u$  satisfies  $R_m$ .

**Proposition 6.1.** *Define the substitution  $\sigma$  by*

$$\begin{aligned}\sigma : 1 &\mapsto 13231 \\ 2 &\mapsto 13231424131 \\ 3 &\mapsto 42324131424 \\ 4 &\mapsto 42324\end{aligned}$$

*Then the fixed point  $\sigma^\infty(1)$  satisfies  $R_4$ .*

*Proof.* By Section 3.1, it is sufficient to consider bispecial factors of  $u = \sigma^\infty(1)$ . The factors of length 2 are  $\mathcal{L}_2(u) = \{13, 14, 23, 24, 31, 32, 41, 42\}$ . For the bispecial factors 1, 2, 23, 2413, the return words are easily determined:

$$\begin{aligned}\mathcal{R}(1) &= \{13, 1323, 1424, 142324\} \\ \mathcal{R}(2) &= \{23, 2314, 2413, 241314\} \\ \mathcal{R}(23) &= \{2314, 2314241314, 232413, 232413142413\} \\ \mathcal{R}(2413) &= \{241314, 24131423, 24132314, 2413231423\}\end{aligned}$$

The language of  $u$  is closed under the morphism  $\varphi$  defined by  $\varphi : 1 \leftrightarrow 4, 2 \leftrightarrow 3$ , since  $\sigma\varphi(w) = \varphi\sigma(w)$  for all factors  $w$ . Therefore we have  $\mathcal{R}(\varphi(w)) = \varphi(\mathcal{R}(w))$ .

The only factors of the form  $a1b$ ,  $a, b \in \mathcal{A}$  are 314 and 413, hence 1 is a weak bispecial factor, and 1, 4 are the only bispecial factors with prefix or suffix 1 or 4. Similarly, 23 and 32 are weak bispecial factors and no other bispecial factor has prefix or suffix 23 or 32.

The return words of the weak bispecial factor 2413142 are factors of  $\sigma(v)$ , with a factor  $v$  of length  $|v| \geq 2$  having prefix 2 or 3, suffix 2 or 3 and no other occurrence of 2 and 3. Since the only possibilities for  $v$  are 23, 2413, 32, 3142, we obtain

$$\mathcal{R}(2413142) = \{24131423, 241314232413231423, 24131424132314, 241314241323142324132314\}.$$

All remaining bispecial factors  $w$  have prefix 24132 or 31423 and suffix 23142 or 32413, and therefore a decomposition  $w = t\sigma(v)t'$  with  $t \in \{24, 31\}$ ,  $t' \in \{1323142, 4232413\}$  and a unique bispecial factor  $v$ . If  $v$  is empty, then we have w.l.o.g.  $w = 241323142$  and

$$\mathcal{R}(w) = \{2413231423, 2413231423241314, 2413231424131423, 24132314241314232413242\}.$$

If  $v$  is not empty, then the uniqueness of  $v$  implies that the set of complete return words of  $w$  is  $t\sigma(\mathcal{R}(v))t'$ . Since  $v$  is shorter than  $w$ , we obtain inductively that all bispecial factors have exactly 4 return words.  $\square$

**6.2. Weak bispecial factors.** The preceding example shows that weak bispecial factors cannot be excluded in words  $u$  satisfying  $R_4$ . Nevertheless, we can show that the existence of a weak bispecial factor imposes strong restrictions on the structure of the word  $u$ .

**Lemma 6.2.** *Let  $w$  be a weak bispecial factor of a word  $u$  satisfying  $R_4$ . Then there exist factors  $w_1, w_2 \in \mathcal{Aw} \cup w\mathcal{A}$  and  $v_1, v_2, v_3, v_4$  such that*

$$(2) \quad \mathcal{R}(w_1) = \{v_1v_3, v_1v_4, v_2v_3, v_2v_4\} \text{ and } \mathcal{R}(w_2) = \{v_3v_1, v_3v_2, v_4v_1, v_4v_2\}.$$

*Proof.* Let  $w$  be a weak bispecial factor. Since the problem is symmetric, assume, w.l.o.g.,  $\#\mathcal{E}_r(w) \geq \#\mathcal{E}_\ell(w)$ . We have three different situations:

- $\#\mathcal{E}_r(w) = 2$ : We have  $\mathcal{E}_r(aw) = 1$  for all  $a \in \mathcal{E}_\ell(w)$ . For both  $b \in \mathcal{E}_r(w)$ , there must be more than one return word of  $w$  starting with  $b$  by Lemma 5.5. Therefore, there exists two return words of  $w$  starting with each  $b$ . Let  $w_1 = wb_1$ ,  $w_2 = wb_2$ .
- $\#\mathcal{E}_r(w) = 3$ : There exists a unique  $b \in \mathcal{E}_r(w)$  such that two return words of  $w$  start with  $b$ . By Lemma 5.5, there exists therefore some  $a \in \mathcal{E}_\ell(w)$  such that  $\#\mathcal{E}_r(aw) \geq 2$ . Since  $w$  is weak bispecial with  $\#\mathcal{E}_r(w) = 3$ , we have  $\#\mathcal{E}_r(aw) = 2$  and  $\mathcal{E}_r(a'w) = 1$  for all other  $a' \in \mathcal{E}_\ell(w)$ . Let  $w_1 = aw$ ,  $w_2 = wb$ .
- $\#\mathcal{E}_r(w) = 4$ : For every  $b \in \mathcal{E}_r(w)$ , there is a unique return word of  $w$  starting with  $b$ . By Lemma 5.5, we have  $a_1, a_2 \in \mathcal{E}_\ell(w)$  with  $\#\mathcal{E}_r(a_iw) = 2$ . Let  $w_1 = a_1w$ ,  $w_2 = a_2w$ .

Consider “complete return words of the set  $\{w_1, w_2\}$ ”: words which have either  $w_1$  or  $w_2$  as prefix, either  $w_1$  or  $w_2$  as suffix, and no other occurrence of  $w_1$  and  $w_2$ . By the definitions of  $w_1$  and  $w_2$ , there are exactly two such words  $v_1w_{i_1}, v_2w_{i_2}$  with prefix  $w_1$  and two words  $v_3w_{i_3}, v_4w_{i_4}$  with prefix  $w_2$ .

If  $i_1 = i_2 = 2$  and  $i_3 = i_4 = 1$ , then  $R_4$  implies that (2) holds.

If  $i_1 = i_2 = 1$ , then  $w_1$  has only the two return words  $v_1, v_2$ . If  $i_2 = i_3 = i_4 = 1$ , then the return words of  $w_1$  are  $v_1v_3, v_1v_4, v_2$ . Similarly,  $i_3 = i_4 = 2$  and  $i_1 = i_2 = i_3 = 2$  are not possible.

The only remaining case is  $i_1 = i_4 = 1$ ,  $i_2 = i_3 = 2$ . Then the return words of  $w_1$  are  $v_1$  and  $v_2v_3^{r_i}v_4$ ,  $i \in \{1, 2, 3\}$ ,  $0 \leq r_1 < r_2 < r_3$ . The return words of  $w_2$  are  $v_3$  and  $v_4v_1^{s_i}v_2$ ,  $i \in \{1, 2, 3\}$ ,  $0 \leq s_1 < s_2 < s_3$ .

The return words of  $v_2w_2$  are therefore of the form  $v_2v_3^{r_i}v_4v_1^{s_j}$ . Let  $S_1$  be the set of these 4 pairs  $(r_i, s_j)$ . Similarly, let  $S_2$  be the set of the 4 pairs  $(s_i, r_j)$  such that  $v_4v_1^{s_i}v_2v_3^{r_j}$  is a return word of  $v_4w_1$ .

We show that there must be some  $i \in \{1, 2, 3\}$  such that  $(r_i, s_2) \in S_1$  and  $(r_i, s_3) \in S_1$ , by considering the return words of  $v_1^{s_2}w_1$  and of  $v_1^{s_2}v_2w_2$ . The return words of  $v_1^{s_2}v_2w_2$  are of the form  $v_1^{s_2}v_2tv_3^{r_i}v_4v_1^{s_j-s_2}$  with  $t \in (v_3^*v_4v_1^{s_1}v_2)^*$ ,  $i \in \{1, 2, 3\}$  and  $j \in \{2, 3\}$ . For these  $t$  and  $r_i$ ,  $v_1^{s_2}v_2tv_3^{r_i}v_4$  is a return word of  $v_1^{s_2}w_1$ . If there was no  $r_i$  with  $(r_i, s_2) \in S_1$  and  $(r_i, s_3) \in S_1$ , then these words would provide 4 different return words of  $v_1^{s_2}w_1$ , which contradicts  $R_4$  since  $v_1$  is another return word.

Similarly, we must have some  $i \in \{1, 2, 3\}$  such that  $(s_i, r_2) \in S_2$  and  $(s_i, r_3) \in S_2$ . By considering the return words of  $v_1^{s_2}w_1$  and  $v_4v_1^{s_2}w_1$ , we obtain as well the existence of some  $i \in \{1, 2, 3\}$  such that  $(r_2, s_i) \in S_1$  and  $(r_3, s_i) \in S_1$ . Finally, we must also have some  $i \in \{1, 2, 3\}$  such that  $(s_2, r_i) \in S_2$  and  $(s_3, r_i) \in S_2$ .

Putting everything together, we have two possibilities for  $S_1$ . Either it contains  $(r_1, s_1)$  and no other  $(r_i, s_j)$  with  $i = 1$  or  $j = 1$ , or  $S_1 = \{(r_1, s_2), (r_1, s_3), (r_2, s_1), (r_3, s_1)\}$ . Analogously,  $S_2$  contains  $(s_1, r_1)$  and no other  $(s_i, r_j)$  with  $i = 1$  or  $j = 1$ , or  $S_2 = \{(s_1, r_2), (s_1, r_3), (s_2, r_1), (s_3, r_1)\}$ .

If  $(r_1, s_1) \in S_1$  and  $(s_1, r_1) \in S_2$ , then  $v_2v_3^{r_1}v_4w_1$  has only one return word,  $v_2v_3^{r_1}v_4v_1^{s_1}$ . If  $(r_1, s_1) \notin S_1$  and  $(s_1, r_1) \notin S_2$ , then  $v_2v_3^{r_1}v_4w_1$  has only two return words,  $v_2v_3^{r_1}v_4v_1^{s_2}$  and  $v_2v_3^{r_1}v_4v_1^{s_3}$ . If  $(r_1, s_1) \in S_1$  and  $(s_1, r_1) \notin S_2$ , then the return words of  $v_2v_3^{r_1}v_4w_1$  are of the form  $v_2v_3^{r_1}v_4v_1^{s_1}v_2v_3^{r_i}v_4v_1^{s_j}$  with  $(r_i, s_j) \in S_1 \setminus \{(r_1, s_1)\}$ , thus there are only three words. Similarly,  $v_4v_1^{s_1}v_2w_2$  has only three return words if  $(r_1, s_1) \notin S_1$  and  $(s_1, r_1) \in S_2$ .

This shows that  $i_1 = i_4 = 1$ ,  $i_2 = i_3 = 2$  is impossible, and the lemma is proved.  $\square$

## 7. WORDS ASSOCIATED WITH $\beta$ -INTEGERS

In this section, we describe a new class of infinite words with the property  $R_m$ . The language of these words is not necessarily closed under reversal.

Consider the fixed point  $u = \sigma^\infty(0)$  of a primitive substitution of the form

$$\begin{aligned}
 \sigma : \quad 0 &\mapsto 0^{t_1}1 \\
 &\quad 1 \mapsto 0^{t_2}2 \\
 &\quad \vdots \\
 m-2 &\mapsto 0^{t_{m-1}}(m-1) \\
 m-1 &\mapsto 0^{t_m}
 \end{aligned} \tag{3}$$



for some integers  $m \geq 2$ ,  $t_1, t_m \geq 1$  and  $t_2, \dots, t_{m-1} \geq 0$ . The incidence matrix of  $\sigma$  is a companion matrix of the polynomial  $x^m - t_1 x^{m-1} - \dots - t_m$ . Let  $\beta > 1$  be the dominant root of this polynomial (the Perron-Frobenius eigenvalue of the matrix). If

$$t_j \cdots t_m \prec t_1 \cdots t_m \quad \text{for all } j \in \{2, \dots, m\},$$

where  $\preceq$  denotes the lexicographic ordering, then  $\sigma$  is a  $\beta$ -substitution and  $\beta$  is a simple Parry number. It is easy to see that  $u$  codes in this case the sequence of distances between consecutive nonnegative  $\beta$ -integers

$$\mathbb{Z}_\beta^+ = \left\{ \sum_{j=0}^J x_j \beta^j \mid J \geq 0, x_j \in \mathbb{Z}, x_j \geq 0, x_j \cdots x_0 \prec t_1 \cdots t_m \text{ for all } j, 0 \leq j \leq J \right\},$$

and a letter  $k$  corresponds to the distance  $t_{k+1}/\beta + \dots + t_m/\beta^{m-k}$ . (0 corresponds to distance 1.)

*Remark.* The most prominent example of a  $\beta$ -substitution is the Fibonacci substitution ( $m = 2$ ,  $t_1 = t_2 = 1$ ), where  $\beta$  is the golden mean.

It is not difficult to show that all prefixes of  $u$  are left special factors, with all  $m$  letters being left extensions (see e.g. Frougny, Masáková and Pelantová [9]). For every factor  $w$ , the tree of return words constructed by the left extensions (see Section 3.2) contains therefore a node with  $m$  children, the shortest prefix of  $u$  having  $w$  as suffix. The word  $u$  is uniformly recurrent since all fixed points of primitive substitutions have this property (Queffélec [12]). Therefore every factor  $w$  has at least  $m$  return words. If there exists a left special factor which is not a prefix of  $u$ , then this factor has more than  $m$  return words. By Proposition 3.1, we obtain the following proposition.

**Proposition 7.1.** *If  $u = \sigma^\infty(0)$  for some substitution  $\sigma$  of the form (3), then it satisfies  $R_m$  if and only if  $C(n) = (m-1)n + 1$  for all  $n \geq 0$ .*

Bernat, Masáková and Pelantová [2] characterized the fixed points of  $\beta$ -substitutions satisfying  $\Delta C(n) = m-1$  for all  $n \geq 0$ . The techniques of their proof can also be used to construct non-prefix left special factors if  $\sigma$  is a substitution of the form (3) which is not a  $\beta$ -substitution, and their conditions can be reformulated as in the following corollary.

**Corollary 7.2.** *If  $u = \sigma^\infty(0)$  for some substitution  $\sigma$  of the form (3), then it has the property  $R_m$  if and only if*

- $t_m = 1$  and
- $t_j \cdots t_{m-1} t_1 \cdots t_{j-1} \preceq t_1 \cdots t_{m-1}$  for all  $j \in \{2, \dots, m-1\}$ .

Note that the language of  $u$  is closed under reversal if and only if  $t_1 = t_2 = \dots = t_{m-1}$ . In this case,  $u$  is an Arnoux-Rauzy word of order  $m$ .

#### ACKNOWLEDGEMENTS

The authors acknowledge financial support by the Czech Science Foundation GA ĆR 201/05/0169 and by the grant LC06002 of the Ministry of Education, Youth and Sports of the Czech Republic.

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